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Embedding Pyramids into 3D Meshes

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The pyramid architecture is a powerful topology in the area of computer vision. On the other hand, the 3D mesh architecture possesses rich topological features which make it suitable for building scalable parallel processor systems. The usefulness of these two architectures has led us to consider the problem of embedding pyramids into 3D meshes, for which we present two solutions. The first solution, termed *natural embedding,* maps a pyramid into a 3D mesh such that each level of the pyramid is mapped to a single level of the 3D mesh. The second solution, termed *multiple embedding,* allows simultaneous embedding of multiple pyramids into a single 3D mesh. The quality of both solutions is evaluated using dilation and expansion measures. Using the multiple embedding, we are able to obtain an average dilation of 1.26 and a near-optimal expansion of 1.12. © 1996 Academic Press, Inc.

1. INTRODUCTION

An embedding of a graph G into a graph H is an injection of the nodes in G to the nodes in H. Since network architectures can be represented by graphs, embedding one architecture into another is essentially a graph-to-graph mapping. The embedding problem has gained considerable importance in the area of parallel processing for many reasons. First, efficient parallel algorithms may exist for some source architecture which suits the needs of these algorithms perfectly, and we may wish to implement them on a target architecture [12]. Second, the proof of embedding for the source architecture is also a proof of all algorithms to be implemented in the target architecture, with the level of efficiency determined by the cost associated with the embedding. Further, since the embedded architecture is usually easier to understand and visualize, it is easier to design algorithms for the simpler architectures and then execute them on the target architecture using the embedding transformation. Much research has been done in embedding including, meshes to hypercubes [2], trees to hypercubes [1, 7], meshes to pyramids [4] and pyramids to hypercubes [8, 14].

In this paper, we consider embedding pyramids onto 3D mesh architectures. The motivation for embedding pyramids into 3D meshes is multifold. First, it is well-known that the pyramid is a promising architecture in image processing and image understanding [11, 13]. The most efficient applications of the pyramid are in the areas of scale-space (or multiresolution) and coarse-to-fine operations. Multiscale or multiresolution image representation is a very powerful tool for analyzing numerous image features at multiple scales [6, 11]. Moreover, pyramid machines are not limited just to image processing tasks. By exploiting the hierarchy inherent in the tree structure of a pyramid, and the parallelism inherent at each level, pyramids can handle various problems in graph theory [9], digital geometry [9], and recursive parallel tasks [5].

The 2D mesh architecture, on the other hand, has had wide availability in the research and commercial community. A natural extension of the 2D mesh is the 3D mesh, which has recently gained marked popularity due to a number of inherent architectural features. These include simple VLSI layout, good scalability, higher bandwidth, and smaller diameter compared to 2D mesh (when they have same link width). In addition, since it has three dimensions, it is capable of modeling many physical world problems more naturally, such as 3D image processing and finite element methods. The advantages and rich topological characteristics of 3D mesh have led to the development of the massively parallel MIT J-Machine [3] and, more recently, the CRAY T3D. With the increasing popularity of the 3D mesh and its potential availability coupled with the suitability of many image processing and computer vision applications on a pyramid, we consider the problem of embedding pyramids into 3D meshes.

The rest of the paper is organized as follows. Section 2 presents basic terminology and notations and discusses the measures used for our embedding of the pyramid into 3D mesh. Section 3 presents a simple embedding scheme denoted as *natural embedding.* Section 4 gives an improved embedding scheme denoted as *multiple embedding.* Section 5 provides some conclusions.

2. NOTATIONS AND TERMINOLOGY

The source and the target architectures can be represented as graphs, the *guest graph* and the *host graph.* The guest graph is denoted by G, with vertex set $V(G)$ and edge set $E(G)$. The host graph is denoted by H, with vertex set $V(H)$ and edge set $E(H)$. Each vertex represents a processing element (PE) and each edge represents a link

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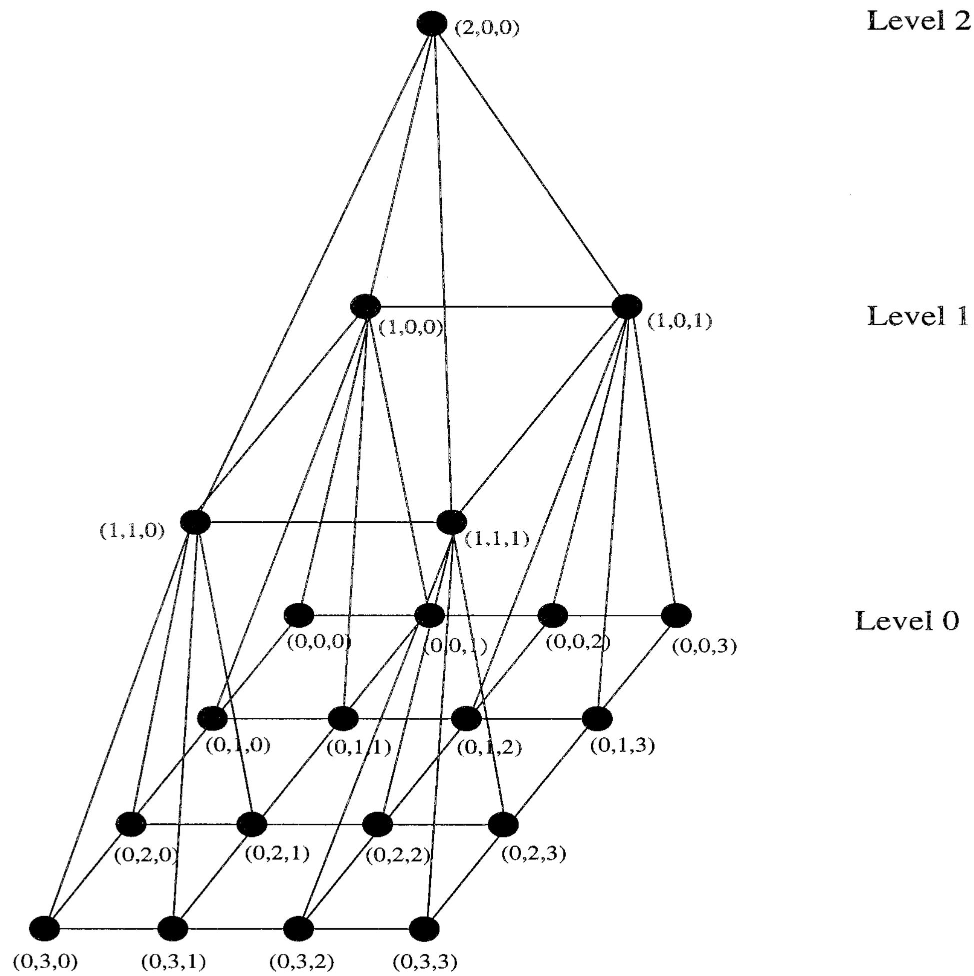

FIG. 1. A pyramid with three levels $P(3)$.

between two PEs. The number of elements in a vertex set is also termed the *size* of the graph. Given G and H, we define a mapping function f for embedding G into H (f: $G \to H$). The function f maps each vertex in G, in a *one-to-one* fashion, to a vertex in H, and each edge in G to a path in H such that if $e = (u, v) \in E(G)$, then $f(e)$ is a path of H with endpoints $f(u)$ and $f(v) \in V(H)$. There are three important measures used for evaluating a mapping which are *dilation, expansion,* and *congestion* [2, 8, 12].

Dilation measures the maximum stretching experienced by any edge in G after embedding. For any edge $e \in E(G)$, the dilation of e under f: $G \to H$ is the length of the path $f(e)$ in H. The *maximum dilation* and the *average dilation* of the mapping function f are defined as

$$\mathrm{Dil}_{\max}(f) = \max_{e \in E(G)} \{\text{length of } f(e)\}.$$

$$\mathrm{Dil}_{\mathrm{ave}}(f) = \mathrm{ave}_{e \in E(G)} \{\text{length of } f(e)\}.$$

Both the maximum and average dilations reflect the degree to which the structure of G is *stretched* by f. Hence, an efficient embedding solution should have low dilation. In the ideal case, the embedding should result in a dilation of 1. We call this ideal case *adjacency preserving* embedding.

Expansion measures the extent of wastage of PEs in the target architecture. Some vertices in H may not have any vertices to be mapped from G, which represent unused PEs in the target architecture. Expansion is simply given by $|V(H)|/|V(G)|$. Expansion gauges how much of H is not directly used in the embedding of G. An expansion of 1 is the ideal case.

Congestion characterizes the traffic flow through the edges of $f(G)$. For $e' \in E(f(G))$, the congestion of e' is $|e \in E(G) : e' \textit{ is in path } f(e)|$. The *maximum congestion* of f is

$$\mathrm{Cong}_{\max}(f) = \max_{e' \in E(f(G))} \{\text{congestion of } e' \text{ under } f\}. \quad (1)$$

Now we define our guest and target architectures. A pyramid can be viewed as a connection of successive levels of smaller 2D square meshes. In order to define a pyramid, we first define a 2D mesh. We denote $M(i)$ to be a $2^i \times$

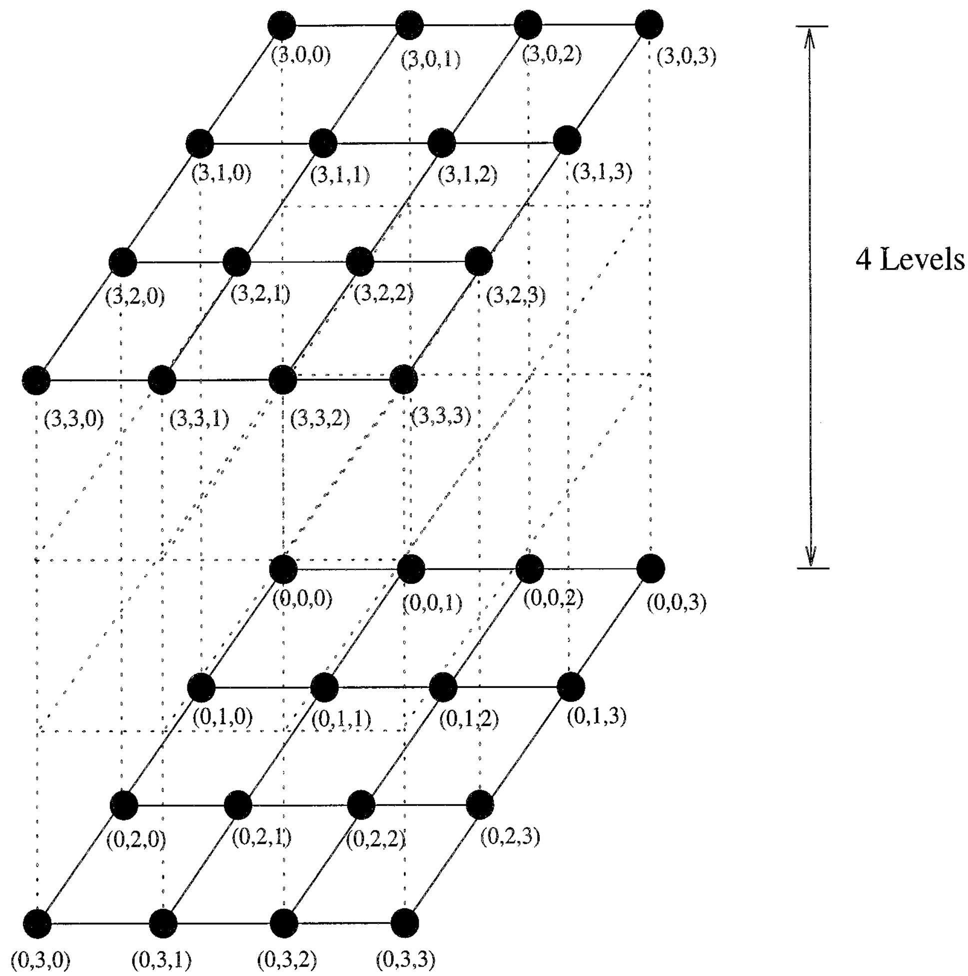

FIG. 2. A 3D mesh with four levels $3DM(4, 4)$.

2^i mesh with a vertex set and an edge set as follows:

$$V(M(i)) = \{(x, y) : 0 \le x, y \le 2^i - 1\},$$

$$E(M(i)) = \{((x_1, y_1), (x_2, y_2)) : (x_1, y_1) \text{ and } (x_2, y_2) \in V(M(i)), \text{ and } |x_1 - x_2| + |y_1 - y_2| = 1\}.$$

With the above graphical representation of a 2D mesh, a pyramid $P(N)$ with N levels can then be defined as having a vertex set and an edge set as follows:

$$V(P(N)) = \{(k, x, y) : (x, y) \in V(M(N - 1 - k)) \text{ and } 0 \le k \le N - 1\},$$

$$E(P(N)) = \{((k, x_1, y_1), (k, x_2, y_2)) : ((x_1, y_1), (x_2, y_2)) \in E(M(N - 1 - k)) \text{ and } 0 \le k \le N - 1\}$$

$$\bigcup \left\{(k, x, y), \left(k + 1, \left\lfloor \frac{x}{2} \right\rfloor, \left\lfloor \frac{y}{2} \right\rfloor\right) : (x, y) \in V(M(N - 1 - k)) \text{ and } 0 \le k \le N - 2\right\}.$$

Here the levels are labeled from 0 to $N - 1$. Figure 1 shows an example of a pyramid of three levels using the above notation.

Similarly, a 3D mesh can be viewed as a connection of successive levels of 2D meshes of size $W \times W$. A 3D mesh of height H, 3DM(W, H), can be defined as having a vertex set and an edge set as follows:

$$V(3DM(W, H)) = \{(k, x, y) : 0 \le x, y \le W - 1 \text{ and } 0 \le k \le H - 1\},$$

$$E(3DM(W, H)) = \{((k, x_1, y_1), (k, x_2, y_2)) : |x_1 - x_2| + |y_1 - y_2| = 1 \text{ and } 0 \le k \le H - 1\} \bigcup \{((k, x, y), (k - 1, x, y)) : 1 \le k \le H - 1\}.$$

Figure 2 shows an example of a 3D mesh with four levels; the levels are labeled from 0 to $H - 1$.

3. NATURAL EMBEDDING

By looking at the structures of the pyramid and 3D mesh, it is evident that a constant dilation cannot be achieved with

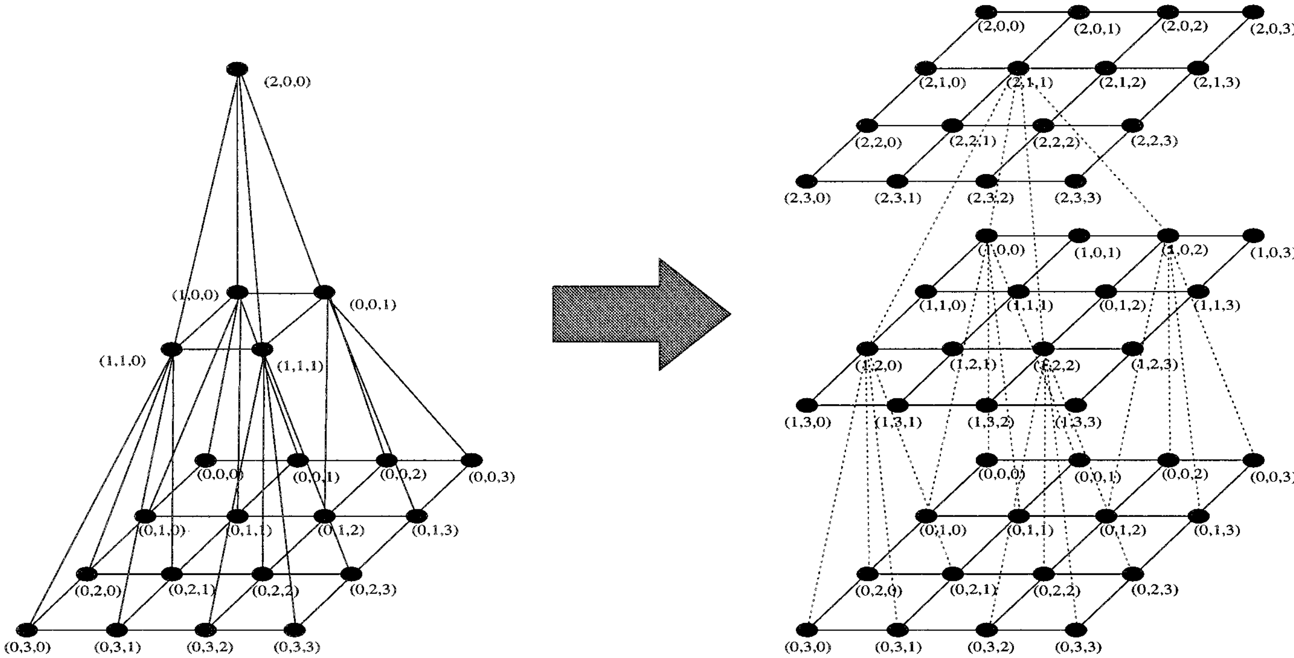

FIG. 3. An example of natural embedding.

any mapping function. This can be seen from the fact that the diameter of a pyramid of size n is $O(\log n)$. On the other hand, a 3D mesh of size n has a diameter of $O(n^{1/3})$. Hence, it must take $O(n^{1/3}/\log n)$ time for the 3D mesh to embed a pyramid structure in an optimal fashion [12]. Thus, the optimal dilation of embedding a pyramid onto a 3D mesh should be $O(n^{1/3}/\log n)$.

3.1. The Mapping Function

We now present our first embedding function denoted *natural embedding.* As its name implies, this embedding maps the pyramid naturally into a 3D mesh. Each level of the pyramid will be mapped to one level of the 3D mesh. As will be shown, the performance of this embedding method is inferior to that of the *multiple embedding* method presented in Section 4. However, it is presented in this paper so that it can be used as a building block for more sophisticated embedding methods such as our multiple embedding method. Moreover, by analyzing the performance measures of the natural embedding, we can determine where improvement is mostly needed when designing a new embedding method. We make the following two assumptions about this embedding solution:

1. The base of the 3D mesh is equal to or larger than the base of the pyramid; i.e., $W \geq 2^{N-1}$.
2. The height of the 3D mesh is equal to or larger than the height of the pyramid; i.e., $H \geq N$.

Let us define the mapping function of natural embedding f as

$$f: V(P(N)) \to V(3DM(W, H)).$$

A node in the pyramid is represented by the coordinates (k, x, y) defined in the 3D space. The mapping function for natural embedding is defined as

$$f(k,x,y) = \begin{cases} (k, x \times 2^k + (2^{k-1} - 1), & \text{if } 1 \leq k \leq N-1 \\ \quad y \times 2^k + (2^{k-1} - 1)) & \\ (k, x, y) & \text{if } k = 0. \end{cases}$$

Figure 3 shows an example of natural embedding of a pyramid with three levels on a 3D mesh of height 3.

When $k = 0$, the nodes of the base of the pyramid will just be mapped to their corresponding counterparts in the 3D mesh. Therefore, the resulting coordinates of a node labeled (k, x, y) in the pyramid will be mapped to the same coordinates (k, x, y) in the 3D mesh. For $1 \leq k \leq N - 1$, i.e., levels from 1 to $N - 1$ of the pyramid, the nodes will be mapped according to the mapping function

$$f(k, x, y) = (k, x \times 2^k + (2^{k-1} - 1), y \times 2^k + (2^{k-1} - 1)).$$

3.2. Embedding Measures

In our natural embedding, the dilation is viewed from two aspects:

1. Dilation within the same levels: it reflects the maximum stretching experienced by an edge in a particular level of the pyramid.
2. Dilation across levels: it reflects the maximum stretching experienced by an edge across two successive levels of the pyramid.

LEMMA 1. *Embedding a pyramid onto a 3D mesh using the natural embedding function f, the edges within level k have dilation 2^k, for $0 \le k \le N-2$.*

Proof. We can show this by using the concept of *city-block distance.* Let (k, x, y) be an arbitrary vertex at level k in the pyramid. Let $NE = \{(k, x-1, y), (k, x, y-1), (k, x+1, y), (k, x, y+1)\}$ be the set of its 4-connected neighbors. Define $CBD(u', v')$ to be the city-block distance between vertices u' and v' in the 3D meshes which is equal to the distance traveled through the k, x and y coordinates while going from u' and v'. It can be shown that the city-block distance between $f(k, x, y)$ and $f(n)$, $CBD(f(k, x, y), f(n))$, is 2^k where $n \in NE$. In this equation, k varies from 0 to $N-2$, as level $N-1$ is the apex in which dilation within level does not apply. ■

LEMMA 2. *Embedding a pyramid onto a 3D mesh using natural embedding function f, the edge between any node at level k and its parent has*

1. *a dilation of $2^k + 1$, for $1 \le k \le N-2$, and*
2. *a maximum dilation of 3 for $k = 0$.*

Proof. (Part 1) Let (k, x, y) be an arbitrary vertex at level k in the pyramid, for $1 \le k \le N-2$. It is connected to its parent $(k+1, \lfloor x/2 \rfloor, \lfloor y/2 \rfloor)$ in the pyramid.

The city-block distance between $f(k, x, y)$ and its parent is given by

$$
\begin{aligned}
&CBD\left(f(k,x,y),\, f\left(k+1, \left\lfloor \frac{x}{2} \right\rfloor, \left\lfloor \frac{y}{2} \right\rfloor\right)\right) \\
&= CBD((k, x \times 2^k + (2^{k-1} - 1), y \times 2^k + (2^{k-1} - 1)), \\
&\qquad \left(k+1, \left\lfloor \frac{x}{2} \right\rfloor \times 2^{k+1} + (2^k - 1), \left\lfloor \frac{y}{2} \right\rfloor \times 2^{k+1} + (2^k - 1)\right) \\
&= 1 + \left| 2^k \times \left(x - 2\left\lfloor \frac{x}{2} \right\rfloor\right) - 2^{k-1} \right| \\
&\quad + \left| 2^k \times \left(y - 2\left\lfloor \frac{y}{2} \right\rfloor\right) - 2^{k-1} \right| \\
&= \begin{cases}
1 + |2^k - 2^{k-1}| + |2^k - 2^{k-1}| & \text{if } x \text{ and } y \text{ are odd} \\
1 + |2^{k-1}| + |2^k - 2^{k-1}| & \text{if } x \text{ is even, } y \text{ is odd} \\
1 + |2^k - 2^{k-1}| + |2^{k-1}| & \text{if } x \text{ is odd, } y \text{ is even} \\
1 + |2^{k-1}| + |2^{k-1}| & \text{if } x \text{ and } y \text{ are even}
\end{cases} \\
&= \qquad 2^k + 1.
\end{aligned}
$$

(Part 2) Let $(0, x, y)$ be an arbitrary vertex at level 0 in the pyramid (the base), connected to its parent $(1, \lfloor x/2 \rfloor, \lfloor y/2 \rfloor)$ in the pyramid.

The city-block distance between $f(0, x, y)$ and its parent is given by

$$
\begin{aligned}
&CBD\left(f(0,x,y),\, f\left(1, \left\lfloor \frac{x}{2} \right\rfloor, \left\lfloor \frac{y}{2} \right\rfloor\right)\right) \\
&= CBD\Big((0, x \times 2^0 + (2^{0-1} - 1), y \times 2^0 + (2^{0-1} - 1)), \\
&\qquad \left(1, \left\lfloor \frac{x}{2} \right\rfloor \times 2^{0+1} + (2^0 - 1), \left\lfloor \frac{y}{2} \right\rfloor \times 2^{0+1} + (2^0 - 1)\right)\Big) \\
&= CBD\left((0, x, y), \left(1, \left\lfloor \frac{x}{2} \right\rfloor \times 2, \left\lfloor \frac{y}{2} \right\rfloor \times 2\right)\right) \\
&= 1 + \left| x - 2\left\lfloor \frac{x}{2} \right\rfloor \right| + \left| y - 2\left\lfloor \frac{y}{2} \right\rfloor \right| \\
&= \begin{cases}
3 & \text{if } x \text{ and } y \text{ are odd} \\
2 & \text{if } x \text{ is even, } y \text{ is odd} \\
2 & \text{if } x \text{ is odd, } y \text{ is even.} \\
1 & \text{if } x \text{ and } y \text{ are even.}
\end{cases}
\end{aligned}
$$

This gives a maximum dilation of 3. ■

The overall maximum dilation always occurs at dilation across levels. In natural embedding, this value occurs when $k = N-2$. This leads to the following theorem.

THEOREM 1. *The overall maximum dilation of embedding a pyramid onto a 3D mesh using natural embedding is equal to $2^{N-2} + 1$.*

The next embedding measure is expansion. In our case, expansion is given by the number of vertices in the 3D mesh divided by the number of vertices in the pyramid:

$$
\begin{aligned}
\text{Number of vertices in 3D mesh} &= 2^{N-1} \times 2^{N-1} \times N \\
&= 4^{N-1} \times N.
\end{aligned}
$$

$$\text{Number of vertices in pyramid} = \frac{(4^N - 1)}{3}.$$

$$\text{Expansion} = \frac{4^{N-1} \times N}{((4^N - 1)/3)} \approx \frac{3}{4} \times N.$$

4. MULTIPLE EMBEDDING

In this section, we present an improved embedding solution and term it the multiple embedding. Since the overall maximum dilation of the natural embedding always occurs at dilation across levels, the objective of the multiple embedding method is to minimize the number of levels needed in the 3D mesh to embed a pyramid. Consequently, this will lead to minimizing the dilation cost as well as the expansion cost as compared to the simple natural embedding. In this embedding, we make the following two assumptions:

1. The number of nodes (PEs) at the base of the 3D mesh is equal to or greater than 1/4 of that of the pyramid; i.e., $W \ge 2^{N-2}$.

2. The number of levels (height) of the 3D mesh is equal to or greater than 6; i.e., $H \ge 6$.

4.1. The Mapping Function

The mapping function m for multiple embedding is defined as follows:

$$m : V(P(N)) \rightarrow V(3DM(W, H))$$

$$m(k,x,y) = \begin{cases} \left(\left|3\left(\left\lfloor\frac{x}{2}\right\rfloor \bmod 2\right) - \left(2(x \bmod 2) + \left(y + \left\lfloor\frac{x}{2}\right\rfloor + \left\lfloor\frac{y}{2}\right\rfloor\right) \bmod 2\right)\right|, \left\lfloor\frac{x}{2}\right\rfloor, \left\lfloor\frac{y}{2}\right\rfloor\right) & \text{if } k = 0 \\ (4, x, y) & \text{if } k = 1 \\ (5, x \times 2^{k-1} + (2^{k-2} - 1), y \times 2^{k-1} + (2^{k-2} - 1)) & \text{if } 2 \le k \le N-1. \end{cases}$$

Using the above function m, the corresponding addresses of the nodes in the pyramid after being mapped to the 3D mesh can be computed in an efficient way. All the nodes in the pyramid are mapped to just six levels of the 3D mesh.

The mapping is divided into three cases as shown in the above mapping function m:

1. Mapping the nodes at level 0 (the base).
2. Mapping the nodes at level 1.
3. Mapping the rest of the nodes above level 1.

Case 1. The mapping function for this case is given by

$$m(k,x,y) = \left(\left|3\left(\left\lfloor\frac{x}{2}\right\rfloor \bmod 2\right) - \left(2\,(x \bmod 2) + \left(y + \left\lfloor\frac{x}{2}\right\rfloor + \left\lfloor\frac{y}{2}\right\rfloor\right) \bmod 2\right)\right|, \left\lfloor\frac{x}{2}\right\rfloor, \left\lfloor\frac{y}{2}\right\rfloor\right),$$
$$\text{where } k = 0.$$

All the nodes at level 0 of the pyramid are mapped to exactly four levels of the 3D mesh. With this approach, dilation 1 can be achieved for 75% of the edges at level 0 of the pyramid. A detailed analysis of the dilation is given in Section 4.2.1. An example of mapping a 4-level pyramid ($N = 4$) onto a 3D mesh is given in Fig. 4. The mapping of the nodes at the base of the pyramid is shown in Fig. 4a.

A group of four nodes is taken into consideration at a time. These four adjacent nodes are mapped to the same (x, y) locations in the 3D mesh but at different levels k in such a manner that a maximum dilation of 1 is maintained with nodes from adjacent groups. There are four cases in mapping a node to a particular level for these four-node groups as shown in Fig. 5.

The mapping of nodes of a group based on the after-mapping location $(\lfloor x/2\rfloor, \lfloor y/2\rfloor)$ of that group is explained as follows:

1. if $\lfloor x/2\rfloor$ and $\lfloor y/2\rfloor$ are odd numbers, then $k = 3 - [2(x \bmod 2) + y \bmod 2]$.
2. if $\lfloor x/2\rfloor$ and $\lfloor y/2\rfloor$ are even numbers, then $k = 2(x \bmod 2) + y \bmod 2$.
3. if $\lfloor x/2\rfloor$ is even and $\lfloor y/2\rfloor$ is odd, then $k = 2(x \bmod 2) + (y + 1) \bmod 2$; or $k = 2(x \bmod 2) + (y + \mathit{anyoddnumber}) \bmod 2$.
4. if $\lfloor x/2\rfloor$ is odd and $\lfloor y/2\rfloor$ is even, then $k = 3 - [2(x \bmod 2) + (y + 1) \bmod 2]$; or $k = 3 - [2(x \bmod 2) + (y + \mathit{anyoddnumber}) \bmod 2]$.

Since $\lfloor x/2\rfloor + \lfloor y/2\rfloor$ will be equal to an odd number if either one and only one of them is odd, we can substitute the above *anyoddnumber* with $\lfloor x/2\rfloor + \lfloor y/2\rfloor$ and combine the above four conditions into the following equation:

$$k = \left(\left|3\left(\left\lfloor\frac{x}{2}\right\rfloor \bmod 2\right) - \left(2(x \bmod 2) + \left(y + \left\lfloor\frac{x}{2}\right\rfloor + \left\lfloor\frac{y}{2}\right\rfloor\right) \bmod 2\right)\right|\right).$$

This algorithm maps all the nodes at level 0 of the pyramid to exactly four levels of 3D mesh.

Case 2. The mapping function for this case is given by

$$m(k, x, y) = (4, x, y), \quad \text{where } k = 1.$$

All the nodes at level 1 ($k = 1$) are mapped to level 4 of the 3D mesh without any change in the (x, y) locations since the number of nodes at level 1 of the pyramid is always equal to or less than that of every level of the 3D mesh and these nodes have the same spatial arrangement. This mapping can be imagined as putting the entire level of nodes on top of the four levels resulted in Case 1, as illustrated in Fig. 4b.

Case 3. The mapping function for this case is given by

$$m(k,x,y) = (5, x \times 2^{k-1} + (2^{k-2} - 1), y \times 2^{k-1} + (2^{k-2} - 1)), \quad \text{where } 2 \le k \le N-1.$$

This case applies to all the nodes located above level 1 of the pyramid. All these nodes are mapped to one single

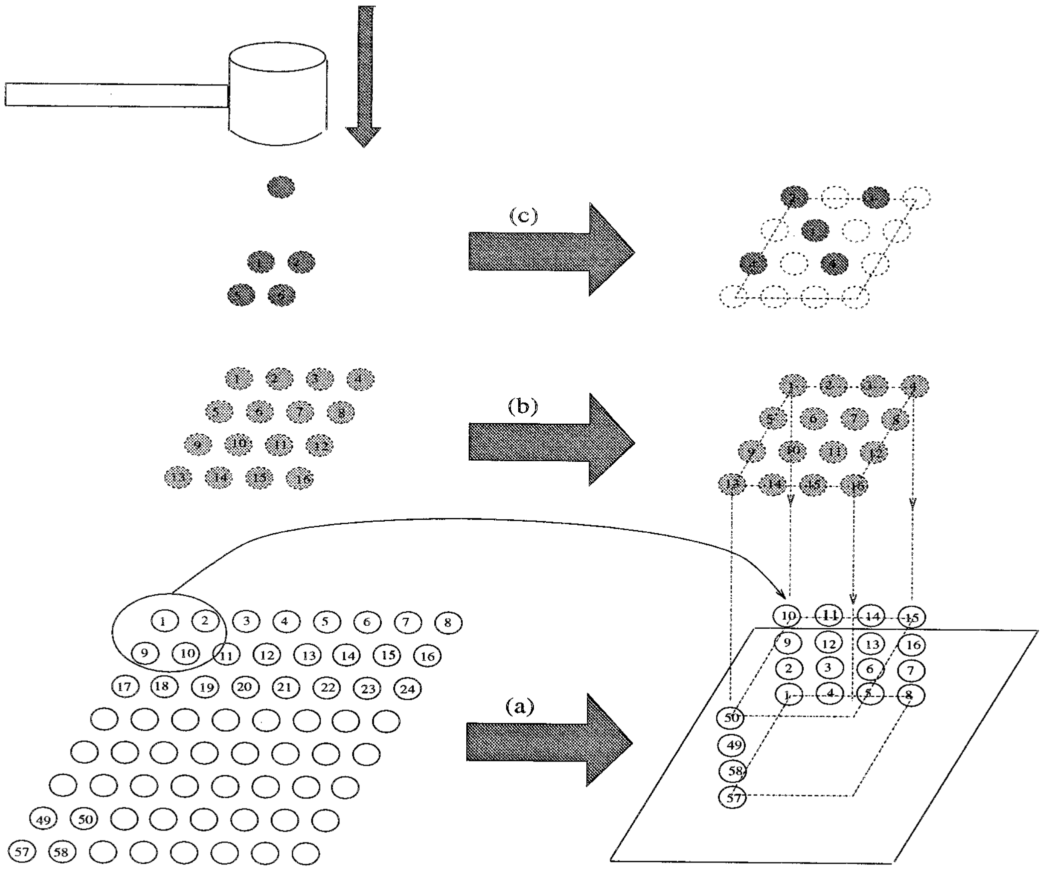

FIG. 4. An example of multiple embedding.

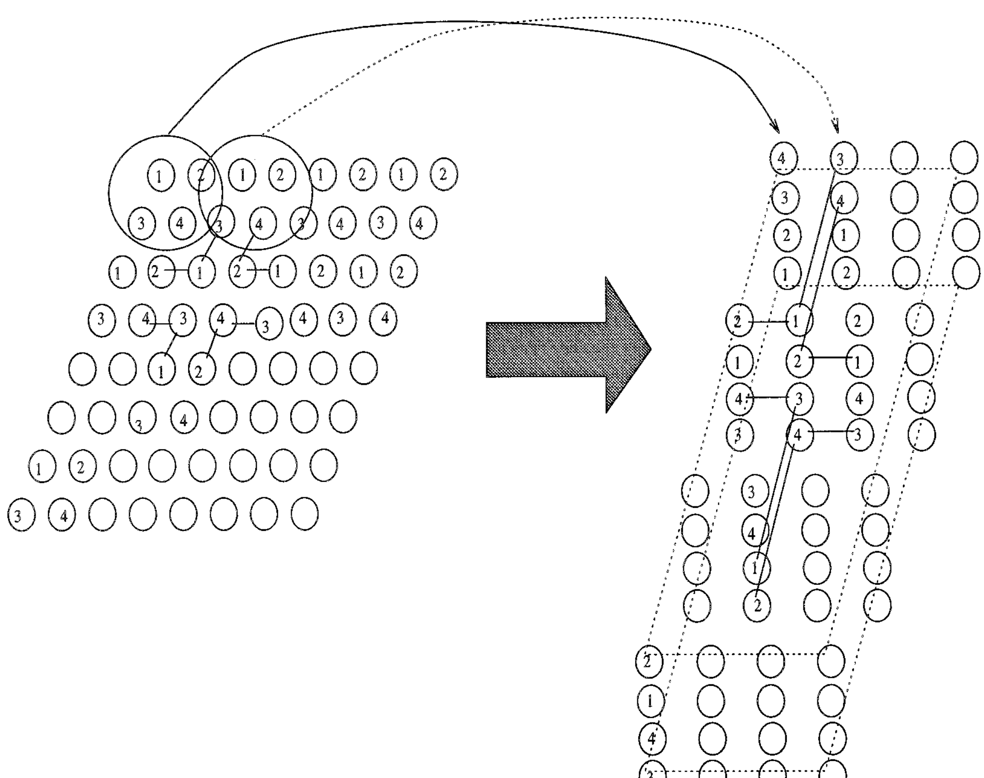

FIG. 5. Multiple embedding: Case 1.

level of the 3D mesh. This mapping is analogous to a *hammering* action, transforming a pyramid into a flat pyramid as illustrated in Fig. 4c. This flattened pyramid forms the sixth level of the 3D mesh after performing the mapping on top of the fifth level resulted from Case 2.

COROLLARY 1. *Given a pyramid with the size of its base = B and a 3D mesh with the size of its base = (1/4) B, all the nodes above level 1 of the pyramid can be mapped to one single level of the 3D mesh.*

Proof. For a pyramid $P(N)$, we can list the number of nodes at each level from the apex to the base level in a sequence 1, 4, 16, 64, ..., 4^{N-1}, which is simply a geometric progression with a common ratio of 4. Let $T(i)$ denote the ith term in this geometric progression and $S(i)$ the sum of the first i terms.

$$T(i) = aR^{i-1}$$

$$S(i) = \frac{a(R^i - 1)}{R - 1}, \quad \text{where } a = 1, R = 4.$$

The number of nodes at level 1 is $T(N-1)$. Since the size of base (level 0) of pyramid $= S$ and

$$\begin{aligned}&\text{Size of base of the 3D mesh}\\&= \frac{1}{4}B = \text{Size of level 1 of the pyramid}\\&= T(N-1) = 4^{N-2}.\end{aligned}$$

$$\begin{aligned}&\text{Number of nodes above level 1 of pyramid}\\&= \text{Sum of nodes from level 2 up to level } N-1\\&= S(N-2) = \frac{(4^{N-2}-1)}{3} < 4^{N-2}\\&< \text{Size of base of the 3D mesh.} \quad \blacksquare\end{aligned}$$

4.2. Embedding Measures

We now discuss the dilation, expansion, and congestion of the multiple embedding function.

4.2.1. Dilation

The dilation can be divided into two cases, namely, within a level and across levels. The dilation within the same level can be further divided into three cases given in the following lemmas.

LEMMA 3. *Using the multiple embedding function m, edges at level* 0 *receive maximum dilation of* 2 *and average dilation of* 5/4.

Proof. Let $(0, x, y)$ denote an arbitrary node at level 0 in the pyramid. We compute the city-block distance between $m(0, x, y)$ and each of its 4-connected neighbors: $m(0, x+1, y)$, $m(0, x-1, y)$, $m(0, x, y+1)$, and $m(0, x, y-1)$ in 3D mesh. These distances are tabulated below.

CBD between $m(0, x, y)$ and	x is even	x is odd
$m(0, x+1, y)$	2	1
$m(0, x-1, y)$	1	2
$m(0, x, y+1)$	1	1
$m(0, x, y-1)$	1	1

Hence, the maximum dilation for all edges at level 0 is 2. The average dilation is roughly 5/4. ■

LEMMA 4. *Using the multiple embedding function m, all edges at level* 1 *have dilation* 1.

Proof. Any node $(1, x, y)$ at level 1 of the pyramid is mapped to $(4, x, y)$ at level 4 of the 3D mesh. The spatial relationship between the nodes is unchanged. Hence, *adjacency preserving* is achieved. ■

LEMMA 5. *Using the multiple embedding m, the edges at level k receive dilation of* 2^{k-1}, $2 \le k \le N-2$.

Proof. Let (k, x, y) be a vertex at level k in the pyramid. Let $NE = \{(k, x-1, y), (k, x, y-1), (k, x+1, y), (k, x, y+1)\}$ be the set of its 4-connected neighbors. Since the mapping function is

$$m(k, x, y) = (5, x \times 2^{k-1} + (2^{k-2} - 1), \; y \times 2^{k-1} + (2^{k-2} - 1)),$$

it can be shown that the city-block distance between $m(k, x, y)$ and $m(n)$,$CBD(m(k, x, y), m(n))$, is 2^{k-1} where $n \in NE$. ■

The dilation across levels can also be divided into three cases given in the following lemmas.

LEMMA 6. *The edge between any node at level* 0 *and its parent has maximum dilation of* 4 *and average dilation of* 2.5.

Proof. Any four nodes at level 0 connected to the same parent at level 1 are mapped to the same (x, y) location at four different levels directly under their parent. Hence, the four edges connecting any four nodes at level 0 to the same parent at level 1 receive respective dilations of 1, 2, 3, 4. Hence, the maximum dilation is 4 and the average dilation is $(1 + 2 + 3 + 4)/4 = 2.5$. ■

LEMMA 7. *The edges between any node at level* 1 *and its parent have maximum dilation of* 3 *and average dilation of* 2.

Proof. Recall that dilation of edges between adjacent nodes at level 1 is 1. The parent of every four children nodes at level 1, after embedding in the 3D mesh, is mapped on top of one of these four children; hence, the four edges connecting any four nodes at level 1 to the same parent at level 2 have dilations of 1, 2, 2, 3, respectively. Let the coordinate of the child directly under the parent in the 3D mesh be $(1, x, y)$. The maximum dilation of 3 occurs between the parent and the farthest child $(1, x + 1, y + 1)$. The average dilation is $(1 + 2 + 2 + 3)/4 = 2$. ■

LEMMA 8. *The edges between any node at level k to its parent have dilation of 2^{k-1}, $2 \le k \le N - 2$.*

Proof. Let (k, x, y) be an arbitrary vertex at level k in the pyramid. It is connected to its parent $(k + 1, \lfloor x/2 \rfloor, \lfloor y/2 \rfloor)$ in the pyramid.

The city-block distance between $f(k, x, y)$ and its parent is given by

$$
\begin{aligned}
&CBD\left(f(k,x,y),\, f\left(k+1, \left\lfloor \frac{x}{2} \right\rfloor, \left\lfloor \frac{y}{2} \right\rfloor\right)\right) \\
&\quad = CBD((5, x \times 2^{k-1} + (2^{k-2} - 1), y \times 2^{k-1} + (2^{k-2} - 1)), \\
&\qquad \left(5, \left\lfloor \frac{x}{2} \right\rfloor \times 2^{k} + (2^{k-1} - 1), \left\lfloor \frac{y}{2} \right\rfloor \times 2^{k} + (2^{k-1} - 1)\right) \\
&\quad = \left| 2^{k-1} \times \left(x - 2\left\lfloor \frac{x}{2} \right\rfloor\right) - 2^{k-2} \right| \\
&\qquad + \left| 2^{k-1} \times \left(y - 2\left\lfloor \frac{y}{2} \right\rfloor\right) - 2^{k-2} \right| \\
&\quad = \begin{cases} |2^{k-1} - 2^{k-2}| + |2^{k-1} - 2^{k-2}| & \text{if } x \text{ and } y \text{ are odd} \\ |2^{k-2}| + |2^{k-1} - 2^{k-2}| & \text{if } x \text{ is even, } y \text{ is odd} \\ |2^{k-1} - 2^{k-2}| + |2^{k-2}| & \text{if } x \text{ is odd, } y \text{ is even} \\ |2^{k-2}| + |2^{k-2}| & \text{if } x \text{ and } y \text{ are even} \end{cases} \\
&\quad = 2^{k-1}. \quad ■
\end{aligned}
$$

The overall maximum dilation occurs across levels. In multiple embedding, this value occurs when $k = N - 2$. This leads to the following theorem.

THEOREM 2. *The overall maximum dilation of embedding a pyramid onto a 3D mesh using multiple embedding is equal to 2^{N-3}.*

4.2.2. Overall Average Dilation

In the above, we have obtained the average dilation both within and across each level of the pyramid. In order to compute an overall average dilation, we have to compute the number of edges within and across each level. Let $e_w(k)$ be number of edges within level k in the pyramid $(0 \le k \le N - 2)$. It can be easily verified that

$$e_w(k) = (2^{N-k-1} - 1) \times 2^{N-k}, \quad 0 \le k \le N - 2.$$

Let $e_a(k)$ be number of edges connecting level k to the above level. It can be easily verified that

$$e_a(k) = 4^{N-k-1}, \quad 0 \le k \le N - 2.$$

Hence, the overall dilation of multiple embedding is given by the following expression:

$$\left\lceil \frac{1.25e_w(0) + e_w(1) + 2.5e_a(0) + 2e_a(1) + \sum_{k=2}^{N-2} (e_w(k) + e_a(k))2^{k-1}}{\sum_{k=0}^{N-2} (e_w(k) + e_a(k))} \right\rceil.$$

To quantify the average dilation obtained in the multiple embedding, we compute the average dilation as a function of N from 3 to 40, that is, for different pyramid sizes. The largest average dilation occurs at $N = 3$, which is just 1.708333, and it decreases with increasing pyramid sizes. When N becomes large, the average dilation converges to 1.265625, which is very close to the optimum dilation of 1. Figure 6 shows a plot of average dilations as a function of the number of pyramid levels.

4.2.3. Expansion

In multiple embedding, only six levels of 3D mesh are required to accommodate all the nodes of the pyramid. Hence, the number of nodes in the 3D mesh can be expressed as follows:

$$
\begin{aligned}
\text{Number of vertices in 3D mesh} &= 2^{N-1} \times 2^{N-1} \times 6 \\
&= 4^{N-1} \times 6.
\end{aligned}
$$

$$\text{Number of vertices in pyramid} = \frac{(4^N - 1)}{3}.$$

$$\text{Expansion} = \frac{4^{N-1} \times 6}{((4^N - 1)/3)} \approx \frac{9}{8}.$$

Since multiple embedding uses only six levels of the 3D mesh, embedding multiple pyramids can be easily achieved which is a very desirable property [10], and the number of such multiple pyramids can be a variable unlike other methods where they have shown how to embed *only* two pyramids into a single hypercube [8]. Furthermore, as shown earlier, the dilation and expansion costs within each of the six levels of the 3D mesh are small.

4.2.4. Congestion

Congestion can also be divided into two cases, namely, within a level and across levels. Congestion within the same level can be further divided into three cases given in the following lemmas.

LEMMA 9. *Using the multiple embedding function m, edges at level 0 of the pyramid have maximum congestion of 2.*

Proof. Recall the mapping of level 0 of the pyramid, every group of four nodes will be mapped one on top of the other in a single column in the 3D mesh. As a result, the maximum congestion is 2 within the group. For nodes that connect with each other in the pyramid but fall into different groups, there will be no edge shared. ■

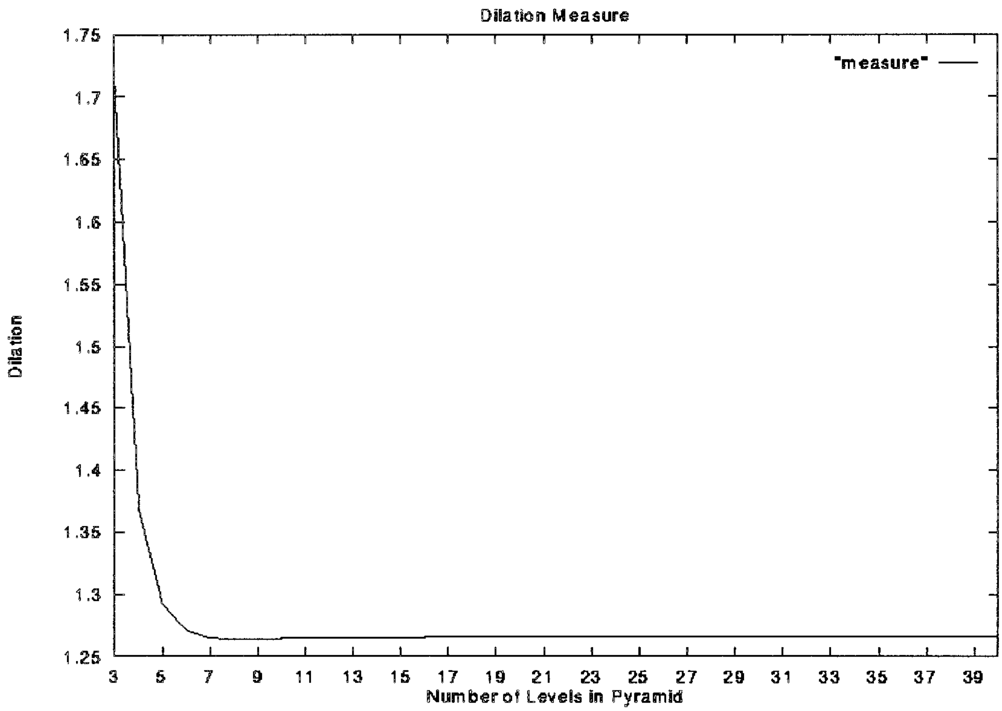

FIG. 6. Dilations for different sizes of pyramids.

LEMMA 10. *Using the multiple embedding function m, edges at level* 1 *have maximum congestion of* 1.

Proof. The proof is obvious since the dilation at level 1 is 1. ■

LEMMA 11. *Using the multiple embedding function m, edges at level* 2 *to level* $N - 1$ *have maximum congestion of* 1.

Proof. Even though nodes from level 2 and above are all mapped to the same level in the 3D mesh, the path between any two nodes in the same level of the pyramid will not have any edge that is shared by a path between two other nodes. This can be depicted clearly in Fig. 7. The lines in this figure represent how nodes are connected with each other. A number k in the figure represents a node in level k in the pyramid. ■

Congestion across levels can also be divided into three cases given in the following lemmas.

LEMMA 12. *The maximum congestion is* 4 *for an edge between any node at level* 0 *of the pyramid and its parent.*

Proof. Any four nodes at level 0 connected to the same parent at level 1 are mapped to the same (x, y) location at four different levels directly under their parent. Hence, the edge between the parent and the child node will have congestion 4. ■

LEMMA 13. *The maximum congestion is* 2 *for nodes at level* 1 *of the pyramid and their parents.*

Proof. The parent of every four nodes of level 1 of the pyramid is mapped on top of one of the four children. Hence, for a child node in level 1 to reach its parent, one edge in the 3D mesh must be shared twice by two of the child nodes before reaching the parent. ■

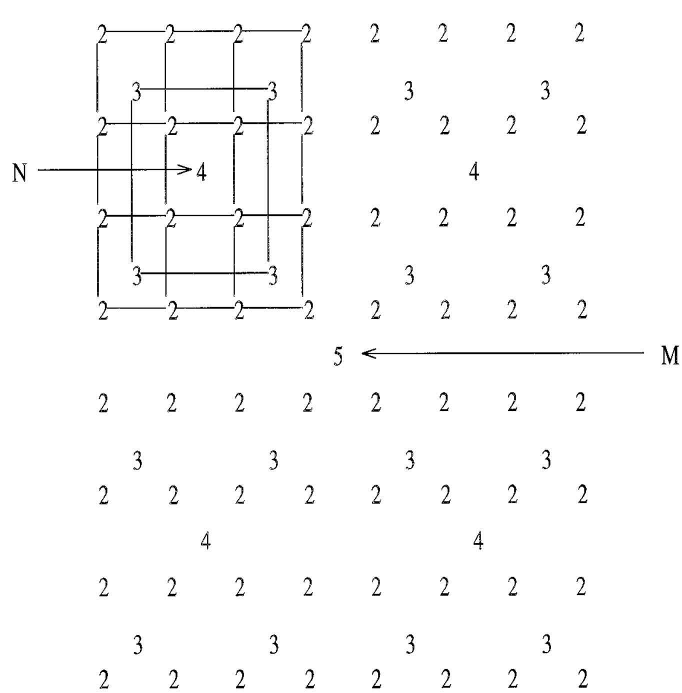

FIG. 7. Nodes communicating within the same level for levels 2 to 5 in P(6).

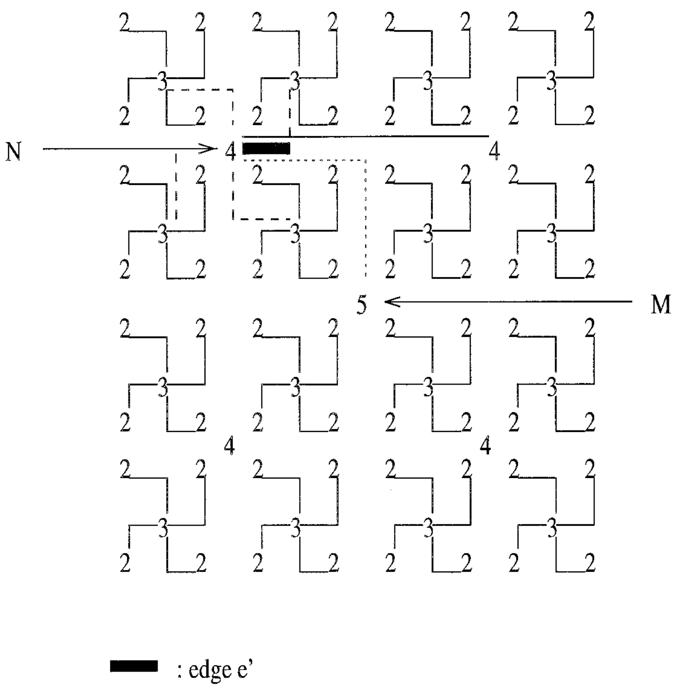

FIG. 8. Nodes communicating across levels for levels 2 to 5 in P(6).

LEMMA 14. *The maximum congestion is* 3 *for nodes at level* 2 *or above of the pyramid.*

Proof. Recall that any node (k, x, y) in levels $2 \leq k \leq N - 1$ is mapped to one single level in the 3D mesh. For a node $N(k, x, y)$ in level 3 or above in the pyramid, in order for such a node to connect to its parent (M), it must travel an edge e' which is already shared by

- N and one of its children (as N is in level 3 or above)
- N and one of its neighbors in the same level.

Thus, e' is shared by a total of three paths, which gives maximum congestion of 3; i.e., the edges with congestion 3 are those connecting a node to its peer, its child, and its parent. Figures 7 and 8 together will give a clear illustration. Note that e' in Fig. 8 is shared by three paths. ■

5. CONCLUSIONS

We have considered the problem of embedding pyramids into 3D meshes. We proposed two solutions: a simple embedding scheme called natural embedding and a more efficient embedding scheme called multiple embedding. Natural embedding maps a pyramid into a 3D mesh naturally with each level of the pyramid mapped to an individual level of the 3D mesh. This mapping results in an overall maximum dilation of $2^{N-2} + 1$ and expansion $\approx (3/4) \times N$ for a pyramid of N levels. In multiple embedding, a 3D mesh of smaller size can be used as the target architecture. This embedding scheme provides a mapping with overall maximum dilation of 2^{N-3} and an average dilation of 1.26. In addition, it has a near-optimal expansion of 1.12 and a maximum congestion of 3. One obvious improvement of the natural embedding method is to relax the assumption that the 3D mesh is of height 6. Rather, we generalize the size of the mesh to be $N \times N \times N$ and the number of pyramid nodes to be as close to N^3 as possible. Then, we investigate its effect on dilation, expansion, and congestion. This is one direction we are currently taking in our research in this area. Finally, it would be useful to experimentally evaluate our embedding schemes.

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